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EXTREMUM PRINCIPLES FOR THE ENERGY-RELEASE PROBLEM OF ELASTIC-PERFECTLY PLASTIC BODY SUBJECTED TO PRESCRIBED CHANGE OF MATERIAL PROPERTIES

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Abstract—Two extremum principles for the energy-release increment in an elastic-perfectly plastic body subjected to prescribed change of the material properties are established. The principles, complementary to each other, can be directly applied to numerical calculations of upper and lower bounds for the energy-release rate problems of fracture mechanics.

1. INTRODUCTION

AN EVALUATION of the energy-release increment in a continuous body resulting from prescribed change of the crack dimensions or shape proved to be a basic problem in fracture mechanics. A spectacular result obtained by Rice[1] and Eshelby[2], who expressed the energy-release rate in terms of the surface integral, has oriented the main effort in this field to the search for more general path independent integrals[3–7].

Whereas much attention has been devoted to the construction of path independent integrals the literature is lacking in investigations into extremum principles corresponding to the energy-release problem which seem to be of major importance for numerical calculations. An exceptional attempt by Bui[5] to develop bounds for the path independent integral in an elastic body is not satisfactory since the bounds for appropriate potentials are not sufficient to evaluate the rate of energy released.

In the present work we use an original concept of Eshelby (see pp. 99–100 of [2]) who derived a path independent integral for a flat crack modeling the crack propagation with the translation of the function of elastic constants in the direction of the crack. The same result was obtained later by Rice[1] with a different method.

Following the concept of Eshelby one can model the formation and/or propagation of a crack of arbitrary shape using two comparison bodies subjected to identical boundary conditions. Namely, we assume that both functions representing the material properties: the free energy function and the dissipation potential are, in general, not continuous in the region occupied by the body and that they can attain the extreme values: zero free energy for every elastic deformation and zero dissipation function for every increment of plastic deformation. The limit values represent a void (or more precisely a zero-strength inclusion) in the body.

Hence general extremum principles derived in the present paper can be in particular used to evaluate the total energy released during the transition from the initial comparison body containing a crack to the final comparison body containing crack of different size or shape. Assuming that both functions representing the material properties are constant within the body excluding the prescribed regions where they vanish we can model a wide range of cracks of different shapes.

A numerical example of the application of the extremum principles to evaluate the total energy released during flat crack propagation in an edge-crack specimen in tension by uniform displacement has been presented in paper[11] which follows the present work. A general result is obtained there for positive-volume void containing the flat crack. The energy bounds for the flat crack are obtained as the limits for the particular case when the void shrinks to the flat crack.

A practical significance of the extremum principles established in Sections 3 and 4 consists in the fact that they make it possible to calculate upper bound and lower bound for the total energy increment with standard (for example finite element) methods of construction of the

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kinematically admissible strain function and statically admissible stress function. The calculation can be performed for a wide class of problems including propagation of crack of arbitrary shape. A unique feature of bounding the energy increment (whereas the principles existing in the literature bound the energy) makes the principles a useful tool to investigate crack problems.

It should be also noted that the basic inequalities expressed in the form of extremum principles were obtained for the assumptions weaker than those commonly used in the literature for the description of the elastic behavior of the body. Namely, we do not assume neither continuity nor differentiability of the free energy function. Using the convex analysis notation, already established in theory of plasticity we obtain a uniform mathematical description of elastic and plastic constitutive laws. In the particular case, when the plastic effect is neglected and the free energy function is differentiable with respect to the strain tensor we arrive at the classical form of the elastic constitutive relations. Indeed, if the function is differentiable then the subdifferential contains exactly one subgradient which is identified with the gradient of the function.

The perfectly-plastic behavior of the body is here determined with a standard concept of plastically admissible region (also called the elastic region) in the space of the stress tensors. The classical inequalities relating the stress and the plastic strain increment are presented (6) in terms of basic notions of convex analysis: the subdifferential and the subgradient. The application of convex analysis in conjunction with the concept of statically admissible and kinematically admissible functions leads directly to final results presented in the paper.

It follows from the above remarks that the problem formulated in the present paper is a natural generalization of the energy-release problem considered in the literature in conjunction with crack propagation. Indeed, the prescribed damage of the material can take the form of 2-dimensional crack, 3-dimensional void where material can not support any forces (zero strength) or arbitrary region where the strength of material is changed.

2. FORMULATION OF THE PROBLEM

We consider an elastic-perfectly plastic body 0 occupying three-dimensional region V bounded by sufficiently regular surface B. The boundary B is decomposed into the surface B_S, where the tractions $\mathbf{T}^B(\mathbf{x})$ are prescribed and the surface B_K, where the displacements $\mathbf{u}^B(\mathbf{x})$ are prescribed.

Assuming that the *plastic strain* function $\boldsymbol{\epsilon}^{0p}(\mathbf{x})$ in the body 0 is given one can determine the *total strain* function $\boldsymbol{\epsilon}^0(\mathbf{x})$ and the *stress* function $\boldsymbol{\sigma}^0(\mathbf{x})$ from the relations

$$\boldsymbol{\sigma}^0(\mathbf{x}) \in \partial W_0(\boldsymbol{\epsilon}^0 - \boldsymbol{\epsilon}^{0p}, \mathbf{x}) \quad \text{in } V \tag{1}$$

$$\epsilon^0_{ij} = \frac{1}{2}(u^0_{i,j} + u^0_{j,i}) \quad \text{and} \quad \sigma^0_{ij,j} = 0 \quad \text{in } V \tag{2}$$

$$u^0_i = u^B_i \quad \text{on} \quad B_K \tag{3}$$

$$T^0_i \equiv \sigma^0_{ij} n_j = T^B_i \quad \text{on } B_S \tag{4}$$

where $W_0(\boldsymbol{\epsilon}, \mathbf{x})$ is the *free energy* function defined for all strain tensors $\boldsymbol{\epsilon}$ and all $\mathbf{x}$ from V and $\mathbf{n}$ is the unit vector normal to the boundary B and taken as positive outwardly. Here it is assumed that the free energy function is lower-semicontinuous [8, 9] and convex with respect to $\boldsymbol{\epsilon}$ and it attains an absolute minimum equal to zero at $\boldsymbol{\epsilon} = \mathbf{0}$. Consequently $\partial W_0(\boldsymbol{\epsilon}^0 - \boldsymbol{\epsilon}^{0p}, \mathbf{x})$ denotes the *subdifferential* (see [8, 9]) of W_0 with respect to $\boldsymbol{\epsilon}$ at $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0(\mathbf{x}) - \boldsymbol{\epsilon}^{0p}(\mathbf{x})$ defined for every $\mathbf{x}$ from V.

Suppose that the considered body changed the material properties while the prescribed boundary conditions remained unchanged. The new material properties are represented by the function $W_1(\boldsymbol{\epsilon}, \mathbf{x})$ defined for every strain tensor $\boldsymbol{\epsilon}$ and every $\mathbf{x}$ from V, which will be referred to as the free energy function of body 1. It is assumed that $W_1(\boldsymbol{\epsilon}, \mathbf{x})$ is lower-semicontinuous and convex with respect to $\boldsymbol{\epsilon}$ and it attains an absolute minimum equal to zero at $\boldsymbol{\epsilon} = \mathbf{0}$.

It is also assumed that the *plastic strain rate* function is constant along the transition path from body 0 to 1

$$\boldsymbol{\epsilon}^{\lambda p}(\mathbf{x}) = \boldsymbol{\epsilon}^{0p}(\mathbf{x}) + \lambda[\boldsymbol{\epsilon}^{1p}(\mathbf{x}) - \boldsymbol{\epsilon}^{0p}(\mathbf{x})] \quad \text{in } V \tag{5}$$

where $0 \leq \lambda \leq 1$ is the parameter of the transition path and $\boldsymbol{\epsilon}^{\lambda p}(\mathbf{x})$ is the actual plastic strain function corresponding to λ. Now, introducing the *dissipation function* $D(\Delta\boldsymbol{\epsilon}, \mathbf{x})$ which represents plastic behavior of the body we can write the *plastic flow law* in the form

$$\boldsymbol{\sigma}^{\lambda}(\mathbf{x}) \in \partial D(\boldsymbol{\epsilon}^{1p} - \boldsymbol{\epsilon}^{0p}, \mathbf{x}) \quad \text{in } V \tag{6}$$

where $\boldsymbol{\sigma}^{\lambda}(\mathbf{x})$ is the actual stress function corresponding to the parameter λ and $\partial D(\boldsymbol{\epsilon}^{1p} - \boldsymbol{\epsilon}^{0p}, \mathbf{x})$ is the subdifferential[8, 9] of the dissipation function at $\Delta\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{1p}(\mathbf{x}) - \boldsymbol{\epsilon}^{0p}(\mathbf{x})$. Here we assume that $D(\Delta\boldsymbol{\epsilon}, \mathbf{x})$ defined for all increments of the plastic strain tensor $\Delta\boldsymbol{\epsilon}$ and for all $\mathbf{x}$ from V is lower-semicontinuous and convex with respect to $\Delta\boldsymbol{\epsilon}$ and it attains an absolute minimum equal to zero at $\Delta\boldsymbol{\epsilon} = \mathbf{0}$.

For the sake of simplicity we shall consider the dissipation function determined by the *plastically admissible region* $E(\mathbf{x})$ prescribed in the space of all stress tensors $\boldsymbol{\sigma}$ for every x from V

$$D(\Delta\boldsymbol{\epsilon}, \mathbf{x}) = \sup_{\boldsymbol{\sigma}} [\Delta\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma} - D^*(\boldsymbol{\sigma}, \mathbf{x})] \quad \text{in } V \tag{7}$$

$$D^*(\boldsymbol{\sigma}, \mathbf{x}) = \begin{cases} 0 & \text{if } \boldsymbol{\sigma} \in E(\mathbf{x}) \\ \infty & \text{otherwise} \end{cases} \tag{8}$$

where the dissipation potential $D^*(\boldsymbol{\sigma}, \mathbf{x})$ is (from the definition 7) the function *polar* to $D(\Delta\boldsymbol{\epsilon}, \mathbf{x})$ (see [8]) and the dot denotes the scalar product. The dissipation potential (8) expressed in terms of the plastically admissible region in the space of stress determines plastic behavior of a large class of elastic-perfectly plastic bodies. In particular the plastically admissible stress region $E(\mathbf{x})$ can be determined with Huber-von Mises or with Treska criterion of plasticity (see [10]).

The strain and stress functions in body 1 satisfy the plastic part of the *constitutive law*

$$\boldsymbol{\sigma}^{1}(\mathbf{x}) \in \partial W_1(\boldsymbol{\epsilon}^{1} - \boldsymbol{\epsilon}^{1p}, \mathbf{x}) \quad \text{in } V \tag{9}$$

as well as the kinematical and statical conditions

$$\epsilon^1_{ij} = \frac{1}{2}(u^1_{i,j} + u^1_{j,i}) \quad \text{and} \quad \sigma^1_{ij,j} = 0 \quad \text{in } V \tag{10}$$

$$u_i^1 = u_i^B \quad \text{on } B_K \tag{11}$$

$$T_i^1 \equiv \sigma^1_{ij} n_j = T_i^B \quad \text{on } B_S. \tag{12}$$

The problem consists in finding the energy release increment ΔE defined by

$$\Delta E = \int_V [W_1(\boldsymbol{\epsilon}^1 - \boldsymbol{\epsilon}^{1p}, \mathbf{x}) - W_0(\boldsymbol{\epsilon}^0 - \boldsymbol{\epsilon}^{0p}, \mathbf{x}) + D(\boldsymbol{\epsilon}^{1p} - \boldsymbol{\epsilon}^{0p}, \mathbf{x})]\, \mathrm{d}V - \int_{B_S} \mathbf{T}^B \cdot (\mathbf{u}^1 - \mathbf{u}^0)\, \mathrm{d}B \tag{13}$$

which represents the difference between the energy stored and dissipated in the body and the energy supplied from outside during the transition from body 0 to body 1.

3. MAXIMUM PRINCIPLE

We introduce the space K of all *kinematically admissible* functions of *strain* $\tilde{\boldsymbol{\epsilon}}(\mathbf{x})$, i.e. the functions which can be derived from the displacement function $\tilde{\mathbf{u}}(\mathbf{x})$; $\tilde{\epsilon}_{ij} = \frac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i})$ in V, satisfying the prescribed boundary condition $\tilde{u}_i = u_i^B$ on B_K.

Consequently we introduce the space S of all *statically admissible stress* functions $\tilde{\sigma}(\mathbf{x})$ which satisfy the equilibrium equation $\tilde{\sigma}_{ij,j} = 0$ in V and the prescribed boundary condition $\tilde{\sigma}_{ij}n_j = T_i^B$ on B_S.

We shall also use the set P of all *plastically admissible stress* functions $\sigma(\mathbf{x})$, i.e. the functions which satisfy $D^*(\sigma, \mathbf{x}) = 0$ in V.

It follows from the elastic part of the constitutive law (1) for body 0 that the integral

$$I_0(\epsilon) = \int_V [W_0(\epsilon, \mathbf{x}) - W_0(\epsilon^0 - \epsilon^{0p}, \mathbf{x}) - (\epsilon - \epsilon^0 + \epsilon^{0p}) \cdot \sigma^0] \, \mathrm{d}V \tag{14}$$

is non-negative for arbitrary strain function $\epsilon(\mathbf{x})$. Similarly, taking into account the particular form (8) of $D^*(\sigma, \mathbf{x})$ we obtain from the plastic part of the constitutive law (6) that

$$G(\Delta\epsilon, \sigma) = \int_V [D(\Delta\epsilon, \mathbf{x}) - \Delta\epsilon \cdot \sigma] \, \mathrm{d}V \geq 0 \tag{15}$$

for arbitrary function $\Delta\epsilon(\mathbf{x})$ provided that the function $\sigma(\mathbf{x})$ is plastically admissible.

It follows from the definition of the *polar function* [8] $W_1^*(\sigma, \mathbf{x})$

$$W_1^*(\sigma, \mathbf{x}) = \sup_{\epsilon} [\epsilon \cdot \sigma - W_1(\epsilon, \mathbf{x})] \quad \text{in } V \tag{16}$$

that the integral

$$U_1(\epsilon, \sigma) = \int_V [W_1(\epsilon, \mathbf{x}) - \epsilon \cdot \sigma + W_1^*(\sigma, \mathbf{x})] \, \mathrm{d}V \tag{17}$$

is non-negative for arbitrary strain function $\epsilon(\mathbf{x})$ and arbitrary stress function $\sigma(x)$.

The above inequalities in conjunction with the identity

$$\begin{aligned} \Delta E = &- \int_V [W_0(\tilde{\epsilon} - \epsilon^{0p}, \mathbf{x}) - (\tilde{\epsilon} - \epsilon^{0p}) \cdot \tilde{\sigma} + W_1^*(\tilde{\sigma}, \mathbf{x})] \, \mathrm{d}V + \int_V (\tilde{\epsilon} - \epsilon^1) \cdot (\sigma^0 - \tilde{\sigma}) \, \mathrm{d}V \\ &+ I_0(\tilde{\epsilon} - \epsilon^{0p}) + U_1(\epsilon^1 - \epsilon^{1p}, \tilde{\sigma}) + G(\epsilon^{1p} - \epsilon^{0p}, \tilde{\sigma}) \end{aligned} \tag{18}$$

obtained from (13) with the divergence theorem, lead directly to the maximum principle:

The function

$$F_1(\tilde{\epsilon}, \tilde{\sigma}) = - \int_V [W_0(\tilde{\epsilon} - \epsilon^{0p}, \mathbf{x}) - (\tilde{\epsilon} - \epsilon^{0p}) \cdot \tilde{\sigma} + W_1^*(\tilde{\sigma}, \mathbf{x})] \, \mathrm{d}V \tag{19}$$

defined for all kinematically admissible strain functions $\tilde{\epsilon}(\mathbf{x}) \in K$, all statically and plastically admissible stress functions $\tilde{\sigma}(\mathbf{x}) \in (S \cap P)$ and the prescribed initial plastic strain function $\epsilon^{0p}(\mathbf{x})$ attains an absolute maximum equal to ΔE at $\tilde{\epsilon} = \epsilon^0$ and $\tilde{\sigma} = \sigma^1$.

The proof follows from the fact that the second term of expression (18) vanishes for arbitrary $\tilde{\sigma}$ from K and $\tilde{\sigma}$ from S and the last three terms are non-negative.

It should be noted that in the considered case the functions $\epsilon^0(\mathbf{x})$, $\epsilon^1(\mathbf{x})$, $\epsilon^{1p}(\mathbf{x})$, $\sigma^0(\mathbf{x})$, $\sigma^1(\mathbf{x})$ which satisfy the relations (1), (4), (6), (10), (11) and (12) are not, in general uniquely determined.

4. MINIMUM PRINCIPLE

To establish the minimum principle we introduce the integral

$$I_0^*(\sigma) = \int_V [W_0^*(\sigma, \mathbf{x}) - W_0^*(\sigma^0, \mathbf{x}) - (\sigma - \sigma^0) \cdot (\epsilon^0 - \epsilon^{0p})] \, \mathrm{d}V \tag{20}$$

which is non-negative for arbitrary stress function $\sigma(x)$ since the function $\sigma^0(x)$ satisfies the constitutive law (1) in body 0. Here $W_0^*(\sigma, x)$ denotes the function polar to $W_0(\epsilon, x)$ (see 16).

Now, taking into account that the functions $\sigma^0(x)$, $\sigma^1(x)$, $\epsilon^0(x)$, $\epsilon^1(x)$, $\epsilon^{1p}(x)$ satisfy the constitutive law (1), (6) (9), we obtain from (13)

$$\Delta E = \int_V [W_1(\bar{\epsilon} - \epsilon^{0p}, x) - (\bar{\epsilon} - \epsilon^{0p}) \cdot \bar{\sigma} + W_0^*(\bar{\sigma}, x)]\,dV + \int_V (\epsilon^0 - \bar{\epsilon}) \cdot (\sigma^1 - \bar{\sigma})\,dV - I_0^*(\bar{\sigma}) - U_1(\bar{\epsilon} - \epsilon^{0p}, \sigma^1) - \int_V D^*(\sigma^1, x)\,dV \quad (21)$$

Hence the minimum principle takes the form:

The function

$$F_u(\bar{\epsilon}, \bar{\sigma}) = \int_V [W_1(\bar{\epsilon} - \epsilon^{0p}, x) - (\bar{\epsilon} - \epsilon^{0p}) \cdot \bar{\sigma} + W_0^*(\bar{\sigma}, x)]\,dV \quad (22)$$

defined for all kinematically admissible strain functions $\bar{\epsilon}(x) \in K$, all statically admissible stress functions $\bar{\sigma}(x) \in S$ and the prescribed initial plastic strain function $\sigma^{0p}(x)$ attains an absolute minimum equal to ΔE at $\bar{\epsilon} = \epsilon^1 - \epsilon^{1p} + \epsilon^{0p}$ and $\bar{\sigma} = \sigma^0$.

The proof follows from the fact that the second term of expression (21) vanishes for arbitrary $\bar{\epsilon}$ from K and $\bar{\sigma}$ from S and the last three terms are non-positive.

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